第 10 章　微分の応用

演習問題 10.1

1.　$y=\sin^{-1}\theta$ から $\theta=\sin y$. $\dfrac{d\theta}{dy}=\cos y$ と

$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\theta^2}$ より

$$\frac{dy}{d\theta}=\frac{1}{\sqrt{1-\theta^2}}$$

2.　$y=\tan^{-1}\theta$ から $\theta=\tan y$. $\dfrac{d\theta}{dy}=\dfrac{1}{\cos^2 y}$ と

$\cos^2 y=\dfrac{1}{1+\tan^2 y}=\dfrac{1}{1+\theta^2}$ より

$$\frac{dy}{d\theta}=\frac{1}{1+\theta^2}$$

3.　①　$z=\omega x+a$ とおくと

$$f'(x)=\frac{d\sin z}{dz}\cdot\frac{d(\omega x+a)}{dx}=\omega\cos(\omega x+a)$$

②　$f'(x)=(\varepsilon^x)'\cos x+\varepsilon^x(\cos x)'=\varepsilon^x(\cos x-\sin x)$

③　$z=(x-1)^2$ とおくと

$$f'(x)=\frac{d\ln z}{dz}\cdot\frac{d(x-1)^2}{dx}=\frac{1}{z}\cdot 2(x-1)=\frac{2}{(x-1)}$$

4.　①　$y=x^{\sin x}$

両辺の対数をとると

$\ln y=\sin x\ln x$

両辺を x で微分すると

$$\frac{y'}{y}=\cos x\ln x+\frac{\sin x}{x}$$

$$f'(x)=y'=x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right)$$

②　$y=(x+4)^3(x+5)^4$ 両辺の対数をとると

$\ln y=\ln[(x+4)^3(x+5)^4]=3\ln(x+4)+4\ln(x+5)$

両辺を x で微分すると

$$\frac{y'}{y}=\frac{3}{x+4}+\frac{4}{x+5}=\frac{7x+31}{(i+4)(x+5)}$$

$$f'(x)=y'=(x+4)^2(x+5)^3(7x+31)$$

演習問題 10.2

1． $q = Ce$ から式(10.12)より

$$i = \frac{dq}{dt} = C\frac{de}{dt} = C\omega E\cos(\omega t + \theta) = C\omega E\sin\left(\omega t + \theta + \frac{\pi}{2}\right)\text{〔A〕}$$

電圧の最下値を $E = E_m$ とすると，電流の最大値は $I_m = C\omega E_m$ となる．

電流は電圧より位相 $\frac{\pi}{2}$ だけ進んでいることがわかる．

2． ① $f'(x) = 1 - 2\sin x$．$f'(x) = 0$ となる x $(0 \leqq x \leqq \pi)$ は $\sin x = \frac{1}{2}$

$x = \frac{6}{\pi}, \ \frac{6}{5}\pi$

$x = \frac{\pi}{6}$ のとき

$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$ （最大値）

$x = \frac{5}{6}\pi$ のとき

$f\left(\frac{5}{6}\pi\right) = \frac{5}{6}\pi - \sqrt{3}$ （最小値）

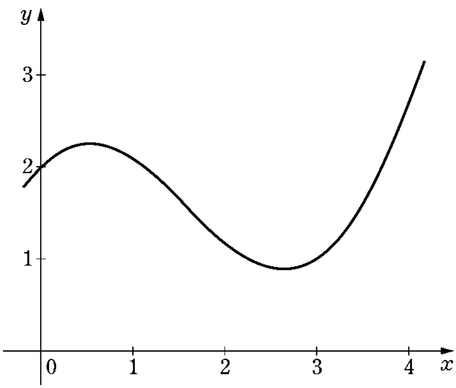
x	0		$\frac{\pi}{6}$		$\frac{5}{6}\pi$		π
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	2	↗	極大	↘	極小	↗	$\pi - 2$

② $f(x) = x\ln x$ ，$\ln x$ から $x > 0$

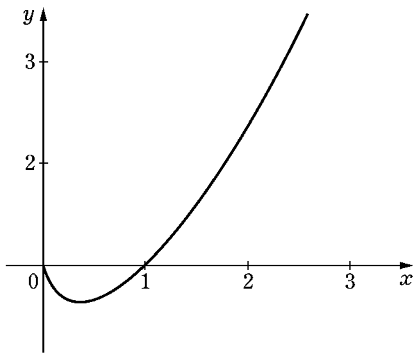
$f'(x) = \ln x + 1$ ，$f'(x) = 0$ となる x の値を $x = \frac{1}{\varepsilon}$

$x = \frac{1}{\varepsilon}$ のとき $f\left(\frac{1}{\varepsilon}\right) = -\frac{1}{\varepsilon}$ （最小値）

x	0		$\frac{1}{\varepsilon}$	
y'		$-$	0	$+$
y		↘	極小	↗



① のグラフ



② のグラフ

演習問題 10.3

1.　①　$f_x=\dfrac{-x^2+y^2}{(x^2+y^2)^2}$ ，$f_y=\dfrac{-2xy}{(x^2+y^2)^2}$

$f_{xx}=\dfrac{2x^3-6xy^2}{(x^2+y^2)^3}$ ，$f_{yy}=\dfrac{-2x^3+6xy^2}{(x^3+y^2)^3}$

$f_{xx}+f_{yy}=0$

②　$f_x=\varepsilon^x(\cos y+\sin y)$ ，$f_y=\varepsilon^x(-\sin y+\cos y)$

$f_{xx}=\varepsilon^x(\cos y+\sin y)$ ，$f_{yy}=\varepsilon^x(-\sin y-\cos y)$

$f_{xx}+f_{yy}=0$

2.　例題 10.10 より

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 f}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial \theta}+\frac{\sin^2\theta}{r}\frac{\partial f}{\partial r}$$

$$+\frac{\sin^2\theta}{r}\frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)+\frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right)$$

$$=\sin^2\theta\frac{\partial^2 f}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial f}{\partial \theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial \theta}+\frac{\cos^2\theta}{r}\frac{\partial f}{\partial r}$$

$$+\frac{\cos^2\theta}{r}\frac{\partial^2 f}{\partial \theta^2}$$

したがって，足し合せると与式を得る．

3.　$t=\dfrac{y}{x}$ とおくと

$$\frac{\partial z}{\partial x}=\frac{\partial f\left(\dfrac{y}{x}\right)}{\partial x}=\frac{\partial f(t)}{\partial t}\cdot\frac{\partial t}{\partial x}=-\frac{y}{x^2}f'(t)$$

$$\frac{\partial z}{\partial y}=\frac{\partial f(t)}{\partial t}\cdot\frac{\partial t}{\partial y}=\frac{1}{x}f'(t)$$

与式に代入すると

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$$

章末問題 10

1. ① $t=x^5+3x^4-2x$ とおくと，$y=(x^5+3x^4-2x)^3=t^3$

$$y'=\frac{dt^3}{dt}\cdot\frac{dt}{dx}=3(5x^4+12x^3-2)(x^5+3x^4-2x)^2$$

② $t=2x^2+1$ とおくと $y=\sqrt{2x^2+1}=\sqrt{t}$

$$y'=\frac{d\sqrt{t}}{dt}\cdot\frac{dt}{dx}=\frac{4x}{2\sqrt{t}}=\frac{2x}{\sqrt{2x^2+1}}$$

③ $t=x+\sqrt{x^2+1}$ とおくと $t'=1+\dfrac{x}{\sqrt{x^2+1}}$

$$y'=(\ln(x+\sqrt{x^2+1}))'=(\ln t)'=\frac{1}{t}\cdot t'=\frac{1}{x+\sqrt{x^2+1}}\cdot\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}$$

④ $y=x^{\frac{1}{x}}$ ，両辺を対数をとって $\ln y=\dfrac{1}{x}\ln x$

それぞれ x で微分すると $\dfrac{y'}{y}=\dfrac{1-\ln x}{x^2}$

$$y'=x^{\frac{1}{x}}\cdot x^{-2}(1-\ln x)=x^{(\frac{1}{x}-2)}(1-\ln x)$$

⑤ $y=\tan^{-1}\dfrac{1-x}{1+x}$, $\left(\dfrac{1-x}{1+x}\right)=\tan y$

両辺を x で微分すると $\left(\dfrac{1-x}{1+x}\right)'=\dfrac{-2}{(1+x)^2}=\dfrac{d\tan y}{dy}\cdot\dfrac{dy}{dx}=\dfrac{1}{\cos^2 y}\dfrac{dy}{dx}$

変形すると，$\dfrac{dy}{dx}=\dfrac{-2}{(1+x)^2}\cos^2 y$.

一方 $1+\tan^2 y=\dfrac{1}{\cos^2 y}=1+\left(\dfrac{1-x}{1+x}\right)^2=\dfrac{2(1+x^2)}{(1+x)^2}$ なので

$$\frac{dy}{dx}=\frac{-2}{(1+x)^2}\cdot\frac{(1+x)^2}{2(1+x^2)}=-\frac{1}{(1+x^2)}$$

2. $y=f(x)$ 上の点 $(a,\ b)$ における接線は
$y-b=f'(a)(x-a)$ と書ける．$f'(x)=2x-3$，$f'(1)=-1$，$f(1)=0$ より
$y=-(x-1)$

3. ① $f_x=k\cos kx$，$f_{xx}=-k^2\sin kx$

② $f_x=\dfrac{1}{2\sqrt{x}}$，$f_{xx}=-\dfrac{1}{4\sqrt{x^3}}$

③ $f_x=\varepsilon^x(x^2+2x)$，$f_{xx}=\varepsilon^x(x^2+4x+2)$

④ $f_x=\varepsilon^{-x}(\cos x-\sin x)$，$f_{xx}=-2\varepsilon^{-x}\cos x$

4. 磁束 $\Phi=Ba^2=\mu_0Ha^2=\mu_0H_0a^2\sin\omega t$〔Wb〕
式(10.14)より起電力は

$$e=N\frac{d\Phi}{dt}=\mu_0NH_0a^2\omega\cos\omega t\ \text{〔V〕}$$

5．$f_x=\dfrac{-x}{\sqrt{(x^2+y^2+z^2)^3}}$，$f_{xx}=\dfrac{2x^2-y^2-z^2}{\sqrt{(x^2+y^2+z^2)^5}}$

x，y，z の対象性から　$f_{yy}=\dfrac{2y^2-x^2-z^2}{\sqrt{(x^2+y^2+z^2)^5}}$

$$f_{zz}=\frac{2z^2-x^2-y^2}{\sqrt{(x^2+y^2+z^2)^5}}$$

したがって　$f_{xx}+f_{yy}+f_{zz}=\dfrac{3}{\sqrt{(x^2+y^2+z^2)^3}}-\dfrac{3(x^2+y^2+z^2)}{\sqrt{(x^2+y^2+z^2)^5}}=0$

6．$\dfrac{\partial f}{\partial u}=\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u}+\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}=\cos\theta\dfrac{\partial f}{\partial x}+\sin\theta\dfrac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}=-\sin\theta\frac{\partial f}{\partial x}+\cos\theta\frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial u^2}=\cos\theta\frac{\partial}{\partial x}\left(\cos\theta\frac{\partial f}{\partial x}+\sin\theta\frac{\partial f}{\partial y}\right)+\sin\theta\frac{\partial}{\partial y}\left(\cos\theta\frac{\partial f}{\partial x}+\sin\theta\frac{\partial f}{\partial y}\right)$$

$$=\cos^2\theta\frac{\partial^2 f}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial v^2}=-\sin\theta\frac{\partial}{\partial x}\left(-\sin\theta\frac{\partial f}{\partial x}+\cos\theta\frac{\partial f}{\partial y}\right)+\cos\theta\frac{\partial}{\partial y}\left(-\sin\theta\frac{\partial f}{\partial x}+\cos\theta\frac{\partial f}{\partial y}\right)$$

$$=\sin^2\theta\frac{\partial^2 f}{\partial x^2}-2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x\partial y}+\cos^2\theta\frac{\partial^2 f}{\partial y^2}$$

したがって，$\dfrac{\partial^2 f}{\partial u^2}+\dfrac{\partial^2 f}{\partial v^2}=\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}$